MATH 105A and 110A Review: Partial derivatives, finding the gradient and Jacobian matrix

Facts to Know:

Let $f : \mathbb{R}^m \to \mathbb{R}$, then gradient of $f(x_1, x_2, \ldots, x_m)$ is

$$\nabla f = \left(\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f, \ldots, \frac{\partial}{\partial x_m} f \right)$$

Let $F : \mathbb{R}^m \to \mathbb{R}^n$ and let

$$F(x_1, \ldots, x_m) = \left(f_1(x_1, \ldots, x_m), f_2(x_1, \ldots, x_m), \ldots\ldots, f_n(x_1, \ldots, x_m) \right)$$

where $f_i : \mathbb{R}^m \to \mathbb{R}$. Then the Jacobian matrix of F is the $n \times m$:

$$\nabla F = \begin{bmatrix} — \nabla f_1 — \\ — \nabla f_2 — \\ \vdots \\ — \nabla f_n — \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1 & \cdots & \frac{\partial}{\partial x_m} f_1 \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f_n & \cdots & \frac{\partial}{\partial x_n} f_n \end{bmatrix}$$

Examples:

1. Find the gradient of $f(x, y, z) = xy^2$.

$f : \mathbb{R}^3 \to \mathbb{R}$

$$\nabla f = (y^2, 2xy, 0)$$

2. Find the Jacobian matrix of $F(x, y, z) = (\overset{f_1}{xy}, \overset{f_2}{yz}, \overset{f_3}{xyz})$ at the point $(1, 2, 3)$.

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\nabla F = \begin{bmatrix} y & x & 0 \\ 0 & z & y \\ yz & xz & xy \end{bmatrix}$$

$$\nabla F(1,2,3) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 6 & 3 & 2 \end{bmatrix}$$

3. Find the Jacobian matrix of $F(x, y, z) = (\overset{f_1}{x^2 + y}, \overset{f_2}{y - z})$.

$$F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$\nabla F = \begin{bmatrix} 2x & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$